International Journal of Mathematics, Statistics and Operations Research
Vol. 3; Number 2; 2023, pp. 235-253
https://doi.org/10.47509/IJMSOR.2023.v03i02.04

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Academic Open Access Publishing
www.arfjournals.com

ON SECOND ORDER GRADIENT LIKE SYSTEM WITH PARTIALLY LOJASIEWICZ GRADIENT NONLINEARITY

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Abstract. We establish the convergence to equilibrium states for global and bounded solutions of some decoupled second order gradient like system with nonlinear damping and nonlinearity which satisfies partially the Lojasiewicz gradient inequality. Moreover, we estimate the rate of convergence and we give a non convergence result.

1. Introduction and main results

In this paper we investigate the long time behaviour, as time goes to infinity, of the trajectories of the following second order gradient like system:

$$\begin{cases} \ddot{U}(t) + \|\dot{U}(t)\|_p^{\alpha}\dot{U}(t) = G(V(t))\nabla F(U(t)), \\ \ddot{V}(t) + \|\dot{V}(t)\|_q^{\alpha}\dot{V}(t) = F(U(t))\nabla G(V(t)), \\ U(0) = U_0, V(0) = V_0, \dot{U}(0) = U_1, \dot{V}(0) = V_1, \\ t \in \mathbb{R}_+, U_0, U_1 \in \mathbb{R}^p \text{ and } V_0, V_1 \in \mathbb{R}^q. \end{cases} \tag{1.1}$$

Here $p, q \in \mathbb{N}^*$, $\alpha \in \mathbb{R}_+$, and $F : \mathbb{R}^p \to \mathbb{R}$, $G : \mathbb{R}^q \to \mathbb{R}$ are functions of class C^2.

If $n \in \mathbb{N}^*$, we denote by $\langle \cdot, \cdot \rangle_n$ the canonical scalar product on $\mathbb{R}^n$ and the application $\| \cdot \|_n$ is its corresponding norm. Whenever A is a matrix in $M_n(\mathbb{R})$, then $\|A\|_{n,n}$ denotes the norm of A which is subordinate to $\| \cdot \|_n$, thus

$$\|A\|_{n,n} = \sup_{\|x\|_n \leq 1} \|Ax\|_n.$$

Let us define also the distance between any two subsets B and D of $\mathbb{R}^n$ by

$$dist_n(B, D) = \inf_{(x,y) \in B \times D} \|x - y\|_n.$$

At first, let us consider the most simple case for the first order gradient systems. Let $N \in \mathbb{N}^*$, we consider the following first order differential system

$$\dot{X}(t) = \nabla H(X(t)), t \in \mathbb{R}^+, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34D05.

Key words and phrases. gradient like system, Lojasiewicz inequality, equilibrium, rate of convergence.

Article History
Received : 11 September 2022; Revised : 22 October 2023; Accepted : 28 October 2023; Published : 26 December 2023

To cite this paper
L. Chergui (2023). On Second order Gradient Like System with Partially Lojasiewicz Gradient Nonlinearity. *International Journal of Mathematics, Statistics and Operations Research.* 3(2), 235-253.

where $H : \mathbb{R}^N \to \mathbb{R}$ is a function of class C^2 and $X(t) \in \mathbb{R}^N$. This type of system (1.2) has been studied earlier in the literature, for its history we refer to [24, 11, 1, 16, 17]. Authors have proved if H is analytic then any global and bounded solution of (1.2) converges, as time goes to infinity, to an equilibrium point which is in

$$S_H = \{u \in \mathbb{R}^N / \nabla H(u) = 0\}.$$

Moreover, they obtained some estimates for the rates of convergence. Recall that the basic argument in the proof of these results relies on the Lojasiewicz gradient inequality, which is known also as the LG inequality.

Theorem 1. *([21, 2]) If $H : \mathbb{R}^N \to \mathbb{R}$ is analytic, then for any $y \in S_H$ there exist $\theta_y \in]0, \frac{1}{2}], \sigma_y > 0$ and $C_y > 0$ such that*

$$\forall x \in \mathbb{R}^N, \|x - y\|_N < \sigma_y \Rightarrow \|\nabla H(x)\|_N \geq C_y |H(x) - H(y)|^{1-\theta_y}.$$

Let us remark that if the LG inequality is satisfied for some $\theta_y \in]0, \frac{1}{2}]$, it is also satisfied for any $\tilde{\theta} \in]0, \theta_y]$ in a possibly sufficiently small ball centered on y and may be another constant C_y. If y is not in S_H, the inequality becomes trivial since H is of class C^1.

After, the most of results concerning the first order gradient system have been extended in [14, 10, 5, 6] for the following second order gradient like system:

$$\ddot{X}(t) + \|\dot{X}(t)\|_N^\alpha \dot{X}(t) = \nabla H(X(t)), t \in \mathbb{R}_+. \tag{1.3}$$

In classical Mechanics, the motion of a system with a finite number of degree of freedom is generally governed by a second order differential system. The above system (1.3) may be seen as a qualitative model for the motion of a material point subject to gravity and some nonlinear damping, constrained to evolve on the graph of H. For this view, we refer to [4, 9] and references therein. For the importance and the applications of this type of system in Optimization and many other related dynamical systems, we refer the reader to [7, 8, 25].

In the case when the damping is linear and H is analytic, the system (1.3) has been studied first by Haraux and Jendoubi in [14]: when ($\alpha = 0$), convergence of all global and bounded solutions of (1.3) is established. In [14], authors estimate also the rate of decaying for such solutions. The situation becomes more difficult when the damping term is nonlinear. In [10], the problem (1.3) was studied in the case $\alpha > 0$ and H is analytic having a uniform Lojasiewicz exponent θ_H which depends only on the function H, see [2, 18, 23]. Precisely, it has been proved in [10] that a weak dissipatin ($\alpha > 0$) can always stabilize global and bounded solution of (1.3). In fact, if $\alpha \in [0, \dfrac{\theta_H}{1-\theta_H}[$ there exists $y \in S_H$ such that

$$\lim_{t \to +\infty} (\|X(t) - y\|_N + \|\dot{X}(t)\|_N = 0.$$

Moreover, we have the following estimate for the rate of convergence

$$\lim_{t\to+\infty}(\|X(t)-y\|_N+\|\dot{X}(t)\|_N \lesssim O(t^{-\frac{\theta_H-(1-\theta_H)\alpha}{1-2\theta_H+(1-\theta_H)\alpha}}).$$

However, all these convergence results can fail whenever the function H is supposed of class C^∞. In [22], Palis and De Melo have proved that there exists a nonlinearity $H\in C^\infty(\mathbb{R}^2,\mathbb{R})$ and a bounded solution of the first order gradient system (1.2) for which the w-limit set is the unit circle of $\mathbb{R}^2$. In [3, 20], the non-convergence result of Palis and De Melo [22] has been extended to the second order problem (1.3) with $\alpha=0$.

Let $N=p+q$, if necessary $\mathbb{R}^N$ is viewed as $\mathbb{R}^p\times\mathbb{R}^q$. We can see clearly that the problem 1.1 is a second order gradient like system associated to the nonlinearity

$$H:\mathbb{R}^N\to\mathbb{R},\ H(u,v)=F(u)G(v).$$

In fact, if $X=\begin{pmatrix}U\\V\end{pmatrix}$ then the system (1.1) can be viewed as the following:

$$\ddot{X}(t)+\begin{pmatrix}\|\dot{U}(t)\|_p^\alpha & 0\\ 0 & \|\dot{V}(t)\|_q^\alpha\end{pmatrix}\dot{X}(t)=\nabla H(X(t)).$$

The system (1.3) can be viewed also as

$$\ddot{X}(t)+\begin{pmatrix}(\|\dot{U}(t)\|_p^2+\|\dot{V}(t)\|_q^2)^{\frac{\alpha}{2}} & 0\\ 0 & (\|\dot{U}(t)\|_p^2+\|\dot{V}(t)\|_q^2)^{\frac{\alpha}{2}}\end{pmatrix}\dot{X}(t)=\nabla H(X(t)).$$

So, it is interesting and useful for us that the nonlinear dissipatin of our problem (1.1) is weaker than that dissipatin considered in (1.3).

In this paper, our aim is to prove that functions such as $H(u,v)=F(u)G(v)$ and which satisfies partially the LG inequality can produce also convergence for global and bounded solutions of (1.1). For that we suppose that F satisfies the LG inequality. Let

$$S_F=\{a\in\mathbb{R}^p/\nabla F(a)=0\},$$

So, for all $a\in S_F$, there exist $\theta_a\in]0,\frac{1}{2}]$, $C_a>0$ and $\sigma_a>0$ such that

$$\forall u\in\mathbb{R}^p,\ \|u-a\|_p<\sigma_a\Rightarrow\|\nabla F(u)\|_p\geq C_a|F(u)-F(a)|^{1-\theta_a}. \tag{1.4}$$

In order to establish the convergence of any global and bounded solution (U,V) of (1.1). It is useful to suppose that F has a uniform Lojasiewicz exponent with respect to the $w-$limit set of the solution (U,V) and not necessary to be uniform with respect to S_F. The following result makes possible the existence of such exponent uniformly on any compact and connected subset of S_F.

Proposition 1. *([5]) Let $L : \mathbb{R}^p \to \mathbb{R}$ be a function of class C^1. Let Γ be a compact connected subset of $S_L = \{x \in \mathbb{R}^p, \nabla L(x) = 0\}$. We assume that* (1.4) *holds for the function L. Then we have:*

- *L takes a constant value L_Γ on Γ.*
- *There exist $\mu > 0, \theta_\Gamma \in]0, \frac{1}{2}]$ and $C_\Gamma > 0$ which are uniform with respect to Γ such that*

$$\forall u \in \mathbb{R}^p, dist_p(u, \Gamma) \leq \mu \Rightarrow \|\nabla L(u)\|_p \geq C_\Gamma |L(u) - L_\Gamma|^{1-\theta_\Gamma}.$$

As in some existing papers on the convergence for gradient like systems, see [14, 10, 5, 6], we restrict our study for the cases in which the system (1.1) admits a strict Lyapunov function. So, in order to develop classical Lojasiewicz-Lyapunov method, it is useful to take a limiting point (a, b) of the solution (U, V) and then we make the change of the variable u by $a + w$. In that case, the function F will be replaced by the function $L(w) = F(a + w) - F(a)$. Hence, the system (1.1) becomes

$$\begin{cases} \ddot{W}(t) + \|\dot{W}(t)\|_p^\alpha \dot{W}(t) = G(V(t))\nabla L(W(t)), \\ \ddot{V}(t) + \|\dot{V}(t)\|_q^\alpha \dot{V}(t) = L(W(t))\nabla G(V(t)) + F(a)\nabla G(V(t)). \end{cases} \tag{1.5}$$

We note the appearance of the term $F(a)\nabla G(V(t))$ in the second differential equation of the new system (1.5). This gradient term requires us to suppose that function G satisfies also the LG inequality near the point b. Let us remark that until now it is unknown if F and G satisfies separately the LG inequality, then convergence for the global and bounded solutions of the problem (1.1) holds. In the sequel, we assume

$$S_F \subset Z_F = \{u \in \mathbb{R}^p / F(u) = 0\}. \tag{1.6}$$

Under this assumption, we have

$$S_F \times \mathbb{R}^q \subset S_H = \{(u, v) \in \mathbb{R}^N / G(v)\nabla F(u) = F(u)\nabla G(v) = 0\}. \tag{1.7}$$

In this paper our preliminary results are the following:

Theorem 2. *For any initial data $(U_0, V_0, U_1, V_1) \in \mathbb{R}^N \times \mathbb{R}^N$, there exists a unique local solution for the second order differential system* (1.1).

Remark 1. *The above Theorem shows the existence of global solution for the Cauchy problem* (1.1)*, at least whenever the initial data is small enough: this is a pure result from the Cauchy-Lipshitz Theorem which is based on fixed Theorem arguments.*

Now, let us define

$$P_u : (u, v) \in \mathbb{R}^N \to u \in \mathbb{R}^p \text{ and } P_v : (u, v) \in \mathbb{R}^N \to v \in \mathbb{R}^q.$$

We define also the w-limit set of any solution (U, V) of the main problem (1.1) by

$$W(U, V) = \{(a, b) \in \mathbb{R}^N \text{ such that } \exists t_n \to +\infty \text{ and } (U(t_n), V(t_n)) \to (a, b)\}.$$

Similarly, we define separately

$$W(U) = \{a \in \mathbb{R}^p \text{ such that } \exists r_n \to +\infty \text{ and } U(r_n) \to a\}.$$

and

$$W(V) = \{b \in \mathbb{R}^q \text{ such that } \exists s_n \to +\infty \text{ and } V(s_n) \to b\}.$$

Hence, we have the following results:

Proposition 2. *For any solution* $(U, V) \in W^{2,\infty}(\mathbb{R}_+, \mathbb{R}^N)$ *of problem* (1.1), *we have*

$$\lim_{t\to+\infty} dist_N((U(t), V(t)), W(U, V)) = 0, \tag{1.8}$$

$$W(U, V) \subset W(U) \times W(V), \tag{1.9}$$

$$P_u(W(U, V)) = W(U) \text{ and } P_v(W(U, V)) = W(V). \tag{1.10}$$

Remark 2. *In general, for second order differential system, the inclusion* (1.9) *may be strict. Indeed, the trajectory* $(U(t) = \cos t, V(t) = \sin t), t \in \mathbb{R}_+$, *is a solution for the following ordinary differential system*

$$\begin{cases} \ddot{U}(t) + \|\dot{U}(t)\|_2^\alpha \dot{U}(t) = -U(t) - V(t), \\ \ddot{V}(t) + \|\dot{V}(t)\|_2^\alpha \dot{V}(t) = U(t) - V(t), \end{cases}$$

and $W(U) \times W(V) = [-1, 1] \times [-1, 1]$, *while* $W(U, V)$ *is the unit circle of* $\mathbb{R}^2$.

Proposition 3. *For any solution* $(U, V) \in W^{2,\infty}(\mathbb{R}_+, \mathbb{R}^N)$ *of problem* (1.1), *we have*

$$\lim_{t\to+\infty} \{\|\dot{U}(t)\|_p + \|\dot{V}(t)\|_q\} = 0, \tag{1.11}$$

(1.12) *The set* $W(U, V)$ *is a compact and connected subset of* S_H,

$$\lim_{t\to+\infty} dist_N((U(t), V(t)), S_H) = 0. \tag{1.13}$$

From (1.13), we deduce that global and bounded solutions of system (1.1) approaches the set S_H as times goes to infinity. The question is then to determine whether or not it actually converges to a point in S_H. The main results of this paper are the following:

Theorem 3. *Assume that* (1.4) *and* (1.6) *hold. Let* $(U, V) \in W^{2,\infty}(\mathbb{R}_+, \mathbb{R}^N)$ *a solution of system* (1.1), *such that for some* $\delta \in]0, 1]$ *and* $T_\delta \in \mathbb{R}_+$, *we have*

$$\forall t \geq T_\delta, \ |G(V(t))| \geq \delta. \tag{1.14}$$

Then

(1.15) *the set* $\Gamma := W(U)$ *is compact connected subset of* S_F.

In addition, if $\dfrac{\alpha}{\alpha+1} < \theta_\Gamma$, *equivalently* $\alpha < \dfrac{\theta_\Gamma}{1-\theta_\Gamma}$, *where* θ_Γ *is given by Proposition 1, then there exists* $(a, b) \in S_F \times \mathbb{R}^q$ *such that*

$$\lim_{t\to+\infty} \{\|\dot{U}(t)\|_p + \|\dot{V}(t)\|_q + \|U(t) - a\|_p + \|V(t) - b\|_q\} = 0.$$

If the function G itself satisfies

$$\exists\delta\in]0,1]/\forall v\in\mathbb{R}^q,|G(v)|\geq\delta. \tag{1.16}$$

Then, the following corollary derives obviously from Theorem 3.

Corollary 1. *Assume that* (1.4), (1.6) *and* (1.16) *hold. Let* $(U,V)\in W^{2,\infty}(\mathbb{R}_+,\mathbb{R}^N)$ *be a solution of system* (1.1). *If* $\alpha<\dfrac{\theta_\Gamma}{1-\theta_\Gamma}$, *then there exists* $(a,b)\in S_F\times\mathbb{R}^q$ *such that*

$$\lim_{t\to+\infty}\{\|\dot U(t)\|_p+\|\dot V(t)\|_q+\|U(t)-a\|_p+\|V(t)-b\|_q\}=0.$$

Therefore, the weak dissipatin of our problem (1.1) does not prevent global and bounded solution to be stabilized under the effect of partially LG restoring term. In the next result, we estimate the rate of convergence.

Theorem 4. *Under assumptions of Theorem 3, there exists a constant* $C>0$ *such that for all* $t\in\mathbb{R}_+$, *we have*

$$\|\dot U(t)\|_p+\|\dot V(t)\|_q+\|U(t)-a\|_p+\|V(t)-b\|_q\leq C(1+t)^{-\dfrac{\theta_\Gamma-(1-\theta_\Gamma)\alpha}{1-2\theta_\Gamma+(1-\theta_\Gamma)\alpha}}.$$

In the sequel, the paper is organized as follows: in section 2 we give proofs of Theorem 2, Proposition 2 and Proposition 3. In section 3 we study the asymptotic behavior for global and bounded solutions of the main problem (1.1). In fact, we give the proof of Theorem 3. Section 4 is devoted to estimate the rate of convergence, for that we give the proof of Theorem 4. In the rest of this paper, we give in section 5 a non convergence result.

2. Preliminary results

2.1. **Proof of Theorem 2.** The system (1.1) is equivalent to the first order differential system $\dot Y(t)=R(Y(t)),t\mathbb{R}_+$, where $Y(t)=(U(t),V(t),\dot U(t),\dot V(t))$ and $R:\mathbb{R}^{2N}\to\mathbb{R}^{2N}$ is the function defined by

$$R(u,v,w,z)=(w,z,G(v)\nabla F(u)-\|w\|_p^\alpha w,F(v)\nabla G(u)-\|z\|_q^\alpha z).$$

Our aim now is to prove that R satisfies the local Lipshitz condition on $\mathbb{R}^{2N}$. Since F and G are functions of class C^2, then $(u,v)\to G(v)\nabla F(u)$ and $(u,v)\to F(u)\nabla G(v)$ are functions of class C^1. So, these functions satisfies the local Lipshitz condition on $\mathbb{R}^p\times\mathbb{R}^q$. In addition, the function $w\to\|w\|_p^\alpha w$ satisfies the Lipshitz condition on any compact subset of $\mathbb{R}^p$ which does not contains $0_{\mathbb{R}^p}$. Fortunately, near $O_{\mathbb{R}^p}$ this function is tangent to zero, hence its differential at $0_{\mathbb{R}^p}$ is zero. Consequently, the function $w\to\|w\|_p^\alpha w$ satisfies the local Lipshitz condition on $\mathbb{R}^p$. Similarly, we have the same result for the function $z\to\|z\|_q^\alpha z$ on $\mathbb{R}^q$. Therefore, we deduce that the function R satisfies the local Lipshitz condition on $\mathbb{R}^{2N}$. Then, by using the Cauchy-Lipshitz Theorem: there is a unique local solution for the problem (1.1) for every initial data $(U_0,V_0,U_1,V_1)\in\mathbb{R}^{2N}$.

2.2. **Proof of Proposition 2.** At first, let us suppose that (1.8) does not happen, then there exist $\varepsilon_0 > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}, t_n \geq n \text{ and } dist_N((U(t_n), V(t_n)), W(U,V)) > \varepsilon_0. \tag{2.1}$$

Knowing that $(U(t_n), V(t_n))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^N$, then there exists a subsequence $(U(t_{\varphi(n)}), V(t_{\varphi(n)}))_{n\in\mathbb{N}}$ which converges to some point $(a,b) \in W(U,V)$. So, when n goes to the infinity in (2.1), we get $dist_N((a,b), W(U,V)) > \varepsilon_0$. This contradicts the fact that $(a,b) \in W(U,V)$. Thus the first assertion (1.8) is proved. Secondly, if we take $(a,b) \in W(U,V)$ then there exists a sequence of time $(t_n)_{n\in\mathbb{N}}$ such that $(U(t_n), V(t_n))_{n\in\mathbb{N}}$ converges to (a,b). Then, separately each one of the two sequences $(U(t_n))_{n\in\mathbb{N}}$ and $(V(t_n))_{n\in\mathbb{N}}$ converges, respectively to a and b. Hence, we have $a \in W(U)$ and $b \in W(V)$ which implies that $(a,b) \in W(U) \times W(V)$ and then assertion (1.9) is proved. Now, we have to prove (1.10). For that, we take for example $a \in W(U)$ then there exists $r_n \to +\infty$ such that $U(r_n) \to a$. Knowing that $(V(r_n))_{n\in\mathbb{N}}$ is a bounded sequence of $\mathbb{R}^q$, there exists then a subsequence $(V(t_{\varphi(n)}))_{n\in\mathbb{N}}$ which converges to some $b \in W(V)$. So, the sequence $(U(t_{\varphi(n)}), V(t_{\varphi(n)}))_{n\in\mathbb{N}}$ converges to (a,b), then $(a,b) \in W(U,V)$ and $a = P_u(a,b)$. This proves that $W(U) \subset P_u(W(U,V))$. Whenever it is obvious that $P_u(W(U,V)) \subset W(U)$, then the equality holds. In a similar way, we prove the second assertion $P_v(W(U,V)) = W(V)$.

2.3. **Proof of Proposition 3.** At the beginning, we are going to prove the assertion (1.11). Let $(U,V) \in W^{2,\infty}(\mathbb{R}_+, \mathbb{R}^N)$ be a solution of the system (1.1). So, we multiply, by mean of the corresponding scalar product, the equations of (1.1) respectively by $\dot{U}(t)$ and $\dot{V}(t)$. Then by integrating over $[0,t]$, we obtain

$$\begin{cases} \dfrac{1}{2}\|\dot{U}(t)\|_p^2 - \dfrac{1}{2}\|U_1\|_p^2 + \displaystyle\int_0^t \|\dot{U}(s)\|_p^{\alpha+2} ds = \int_0^t G(V(s))(F(U(s)))' ds, \\ \dfrac{1}{2}\|\dot{V}(t)\|_q^2 - \dfrac{1}{2}\|V_1\|_q^2 + \displaystyle\int_0^t \|\dot{V}(s)\|_q^{\alpha+2} ds = \int_0^t F(U(s))(G(V(s)))' ds. \end{cases} \tag{2.2}$$

By the formula $(FG)' = GF' + FG'$ and equations (2.2), we get

$$\begin{aligned} \int_0^t \{\|\dot{U}(s)\|_p^{\alpha+2} + \|\dot{V}(s)\|_q^{\alpha+2}\} ds =& [G(V(s))F(U(s))]_0^t + \frac{1}{2}\|U_1\|_p^2 + \frac{1}{2}\|V_1\|_q^2 \\ & - \frac{1}{2}\|\dot{U}(t)\|_p^2 - \frac{1}{2}\|\dot{V}(t)\|_q^2. \end{aligned}$$

This implies

$$t \to \|\dot{U}(t)\|_p^{\alpha+2} + \|\dot{V}(t)\|_q^{\alpha+2} \in L^1(\mathbb{R}_+).$$

Using the fact that $(\ddot{U}, \ddot{V}) \in L^\infty(\mathbb{R}_+, \mathbb{R}^N)$, we get $t \to \|\dot{U}(t)\|_p^{\alpha+2} + \|\dot{V}(t)\|_q^{\alpha+2}$ is uniformly continuous on $\mathbb{R}_+$, obviously (1.11) holds.
Now, in order to prove the assertion (1.12), we recall that

$$W(U,V) = \bigcap_{s>0} \overline{\bigcup_{t\geq s} \{(U(t), V(t))\}}.$$

This implies that the set $W(U,V)$ is a compact and connected subset of $\mathbb{R}^N$, see [13] for a simple proof. Next, let $(a,b) \in W(U,V)$ and $t_n \to +\infty$ such that $U(t_n) \to a$ and $V(t_n) \to b$. Writing

$$\begin{aligned} G(b)\nabla F(a) &= \int_0^1 G(b)\nabla F(a)ds \\ &= \lim_{n\to+\infty} \int_0^1 G(V(t_n+s))\nabla F(U(t_n+s))ds \\ &= \lim_{n\to+\infty} \int_0^1 (-\ddot{U} - \|\dot{U}\|_p^\alpha \dot{U})(t_n+s)ds \\ &= \lim_{n\to+\infty} \{-\dot{U}(t_n+1) + \dot{U}(t_n) - \int_{t_n}^{t_n+1} \|\dot{U}\|_p^\alpha \dot{U}(s)ds\} \\ &= 0. \end{aligned}$$

Simultaneously and in the same way we obtain $F(a)\nabla G(b) = 0$. Then

$$(a,b) \in W(U,V) \Rightarrow \begin{cases} G(b)\nabla F(a) = 0 \\ F(a)\nabla G(b) = 0. \end{cases} \tag{2.3}$$

Hence, the w-limit set $W(U,V)$ is a subset of S_H. This complete the proof of (1.12). It remains now to prove the last assertion (1.13). Since we have $W(U,V) \subset S_H$, then

$$dist_N((U(t),V(t)),S_H) \leq dist_N((U(t),V(t)),W(U,V)).$$

Let t goes to infinity, we get

$$\lim_{t\to\infty} dist_N((U(t),V(t)),S_H) = 0, \tag{2.4}$$

and then (1.13) is proved.

3. Convergence result: Proof of Theorem 3

First, let us prove the assertion (1.15). Writing

$$W(U) = \bigcap_{s>0} \overline{\bigcup_{t\geq s} \{(U(t))\}}.$$

Then, the set $\Gamma := W(U)$ is a compact and connected subset of $\mathbb{R}^p$, see [13]. By using (1.14), we have

$$(a,b) \in W(U,V) \Rightarrow |G(b)| \geq \delta > 0.$$

Together with (2.3), implies that

$$W(U,V) \subset S_F \times \mathbb{R}^q.$$

When combined with $P_u(W(U,V)) = W(U)$, see (1.10), this yields

$$\Gamma = W(U) \subset S_F.$$

Thus, assertion (1.15) is proved.
In the rest of this section, we are going to prove the convergence part of our Theorem 3. From (1.4), the function F satisfies the LG inequality. Since that Γ is a compact and connected subset of S_H, then by applying Proposition 1, we get: there exist some real constants $\mu > 0, C_\Gamma > 0, \theta_\Gamma \in]0, \frac{1}{2}]$ and F_Γ such that

$$\forall u \in \mathbb{R}^p, dist_p(u, \Gamma) \leq \mu \Rightarrow \|\nabla F(u)\|_p \geq C_\Gamma |F(u) - F_\Gamma|^{1-\theta_\Gamma}.$$

From assumption (1.6), we have $F_\Gamma = 0$. Let $\theta := \theta_\Gamma$, the previous inequality becomes

$$\forall u \in \mathbb{R}^p, dist_p(u, \Gamma) \leq \mu \Rightarrow \|\nabla F(u)\|_p \geq C_\Gamma |F(u)|^{1-\theta}. \tag{3.1}$$

Once again, by using (1.10), we have

$$dist_N((u,v), W(U,V)) \leq \mu \Rightarrow dist_p(u, \Gamma) \leq \mu.$$

Combining with (3.1), it follows that

$$\forall (u,v) \in \mathbb{R}^N, dist_N((u,v), W(U,V)) \leq \mu \Rightarrow \|\nabla F(u)\|_p \geq C_\Gamma |F(u)|^{1-\theta}. \tag{3.2}$$

Since we have (1.14), from (1.8) and (1.11) there exists $T \geq T_\delta$ such that for all $t \geq T$, we have

$$dist_N((U(t), V(t)), W(U,V)) \leq \mu, \|\dot{U}(t)\|_p \leq 1, \|\dot{V}(t)\|_q \leq 1 \text{ and } |G(V(t))| \geq \delta. \tag{3.3}$$

So, from (3.2) we get also

$$\forall t \geq T, \|\nabla F(u)\|_p \geq C_\Gamma |F(u)|^{1-\theta}. \tag{3.4}$$

The previous inequality is crucial in what follows. At this step, let ε be a positive real number which will be fixed later and define for all $t \in \mathbb{R}_+$ the functions

$$E(t) = \frac{1}{2}\|\dot{U}(t)\|_p^2 + \frac{1}{2}\|\dot{V}(t)\|_q^2 - F(U(t))G(V(t)),$$

and

$$K(t) = E(t) - \varepsilon \|G(V(t))\nabla F(U(t))\|_p^\alpha \langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p.$$

Our aim now is to prove that K is a strict Lyapunov function, that means $K(t)$ is non increasing and the solution $(U(t), V(t))$ will be constant if $K(t)$ vanishes at some t. By differentiating E, we obtain for all $t \in \mathbb{R}_+$

$$\begin{aligned}
E'(t) =& \langle \ddot{U}(t), \dot{U}(t)\rangle_p + \langle \ddot{V}(t), \dot{V}(t)\rangle_q - \langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p \\
& - \langle F(U(t))\nabla G(V(t)), \dot{V}(t)\rangle_q. \\
=& \langle \ddot{U}(t) - G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p + \langle \ddot{V}(t) - F(U(t))\nabla G(V(t)), \dot{V}(t)\rangle_q \\
=& \langle -\|\dot{U}(t)\|_p^\alpha \dot{U}(t), \dot{U}(t)\rangle_p + \langle -\|\dot{V}(t)\|_p^\alpha \dot{V}(t), \dot{V}(t)\rangle_q \\
=& - \|\dot{U}(t)\|_p^{\alpha+2} - \|\dot{V}(t)\|_q^{\alpha+2}.
\end{aligned}$$

Now, we will differentiate the mixed term

$$M(t) := \|G(V(t))\nabla F(U(t))\|_p^\alpha \langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p.$$

In the next, the calculation is valid only at those points where $G(V)\nabla F(U)$ is not zero. So, we have

$$\begin{aligned}M'(t) =&\alpha\|G(V(t))\nabla F(U(t))\|_p^{\alpha-2}\langle (G(V(t))\nabla F(U(t)))', G(V(t))\nabla F(U(t))\rangle_p\\ &\langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p + \|G(V(t))\nabla F(U(t))\|_p^\alpha \langle G(V(t))\nabla F(U(t)), \ddot{U}(t)\rangle_p\\ &+ \|G(V(t))\nabla F(U(t))\|_p^\alpha \langle (G(V(t))\nabla F(U(t)))', \dot{U}(t)\rangle_p.\end{aligned}$$

Since we have

$$\ddot{U}(t) = G(V(t))\nabla F(U(t)) - \|\dot{U}(t)\|_p^\alpha \dot{U}(t),$$

and

$$(G(V(t))\nabla F(U(t)))' = \langle \nabla G(V(t)), \dot{V}(t)\rangle_q \nabla F(U(t)) + G(V(t))\nabla^2 F(U(t)).\dot{U}(t).$$

Then, we get

$$\begin{aligned}M'(t) =&\alpha\|G(V(t))\nabla F(U(t))\|_p^{\alpha-2}\langle \nabla G(V(t)), \dot{V}(t)\rangle_q \langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p\\ &\langle G(V(t))\nabla F(U(t)), \nabla F(U(t))\rangle_p + \alpha\|G(V(t))\nabla F(U(t))\|_p^{\alpha-2}\\ &\langle G(V(t))\nabla F(U(t)), G(V(t))\nabla^2 F(U(t)).\dot{U}(t)\rangle_p \langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p\\ &+ \|G(V(t))\nabla F(U(t))\|_p^\alpha \langle \nabla G(V(t)), \dot{V}(t)\rangle_q \langle \nabla F(U(t)), \dot{U}(t)\rangle_P\\ &+ G(V(t))\|G(V(t))\nabla F(U(t))\|_p^\alpha \langle \nabla^2 F(U(t)).\dot{U}(t), \dot{U}(t)\rangle_p\\ &+ \|G(V(t))\nabla F(U(t))\|_p^{\alpha+2} - G(V(t))\|G(V(t))\nabla F(U(t))\|_p^\alpha \|\dot{U}(t)\|_p^\alpha\\ &\langle \nabla F(U(t)), \dot{U}(t)\rangle_p.\end{aligned}$$

Replacing in $K'(t)$. We obtain

$$\begin{aligned}K'(t) =& -\|\dot{U}(t)\|_p^{\alpha+2} - \|\dot{V}(t)\|_q^{\alpha+2} - \varepsilon\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2}\\ &+ \varepsilon G(V(t))\|G(V(t))\nabla F(U(t))\|_p^\alpha \|\dot{U}(t)\|_p^\alpha \langle \nabla F(U(t)), \dot{U}(t)\rangle_p\\ &- \varepsilon\|G(V(t))\nabla F(U(t))\|_p^\alpha \langle \nabla G(V(t)), \dot{V}(t)\rangle_q \langle \nabla F(U(t)), \dot{U}(t)\rangle_p\\ &- \varepsilon G(V(t))\|G(V(t))\nabla F(U(t))\|_p^\alpha \langle \nabla^2 F(U(t)).\dot{U}(t), \dot{U}(t)\rangle_p\\ &- \varepsilon\alpha\|G(V(t))\nabla F(U(t))\|_p^{\alpha-2}\langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p \langle \nabla G(V(t)), \dot{V}(t)\rangle_q\\ &\langle \nabla F(U(t)), G(V(t))\nabla F(U(t))\rangle_p - \varepsilon\alpha\|G(V(t))\nabla F(U(t))\|_p^{\alpha-2}\\ &\langle G(V(t))\nabla F(U(t)), \dot{U}(t)\rangle_p \langle \nabla^2 F(U(t)).\dot{U}(t), G(V(t))\nabla F(U(t))\rangle_p.\end{aligned}$$

Let

$$C_1 = \sup_{t\in\mathbb{R}_+} \|\nabla F(U(t))\|_p, C_2 = \sup_{t\in\mathbb{R}_+} \|\nabla G(V(t))\|_p,$$
$$C_3 = \sup_{t\in\mathbb{R}_+} \|\nabla^2 F(U(t))\|_{p,p} \text{ and } C_4 = \sup_{t\in\mathbb{R}_+} |G(V(t))|.$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}K'(t) \leq& -\|\dot{U}(t)\|_p^{\alpha+2} - \|\dot{V}(t)\|_q^{\alpha+2} - \varepsilon\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2}\\&+ \frac{\varepsilon}{2}\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\|\dot{U}(t)\|_p^{\alpha}\{\|G(V(t))\nabla F(U(t))\|_p^2 + \|\dot{U}(t)\|_p^2\}\\&+ \frac{\varepsilon}{2}C_1C_2\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\{\|\dot{U}(t)\|_p^2 + \|\dot{V}(t)\|_q^2\}\\&+ \varepsilon C_3C_4\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\|\dot{U}(t)\|_p^2\\&+ \frac{\varepsilon\alpha}{2}C_1C_2\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\{\|\dot{U}(t)\|_p^2 + \|\dot{V}(t)\|_q^2\}\\&+ \varepsilon\alpha C_3\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\|\dot{U}(t)\|_p^2.\end{aligned}$$

As we saw, the calculation is valid only at those points where $G(V)\nabla F(U)$ is not zero. However, near a point where $G(V)\nabla F(U) = 0$ the worst term

$$\|G(V)\nabla F(U)\|_p^{\alpha}\langle G(V)\nabla F(U), \dot{U}\rangle_p$$

is tangent to zero, so its derivative is zero and the same final estimate holds true. Our aim now is to prove that K is a non increasing function on the interval $[T, +\infty[$. From (3.3), we have

$$\forall t \geq T, \|\dot{U}(t)\|_p \leq 1 \text{ and } \|\dot{V}(t)\|_q \leq 1. \tag{3.5}$$

Then, we have for all $t \geq T$

$$\begin{aligned}K'(t) \leq& -\|\dot{U}(t)\|_p^{\alpha+2} - \|\dot{V}(t)\|_q^{\alpha+2} - \frac{\varepsilon}{2}\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2}\\&+ \frac{\varepsilon}{2}C_5\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\|\dot{U}(t)\|_p^2 + \frac{\varepsilon}{2}C_6\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\|\dot{V}(t)\|_q^2.\end{aligned}$$

Here $C_5 = 1 + (1+\alpha)C_1C_2 + 2C_3C_4 + 2\alpha C_3$ and $C_6 = (1+\alpha)C_1C_2$.
Using Young's inequality, we get

$$\begin{aligned}\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\|\dot{U}(t)\|_p^2 \leq& \frac{1}{4C_5}\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2} + (4C_5)^{\frac{\alpha}{2}}\|\dot{U}(t)\|_p^{\alpha+2},\\\|G(V(t))\nabla F(U(t))\|_p^{\alpha}\|\dot{V}(t)\|_q^2 \leq& \frac{1}{4C_6}\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2} + (4C_6)^{\frac{\alpha}{2}}\|\dot{V}(t)\|_q^{\alpha+2}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}K'(t) \leq& \{-1 + \frac{\varepsilon}{2}C_5(4C_5)^{\frac{\alpha}{2}}\}\|\dot{U}(t)\|_p^{\alpha+2} + \{-1 + \frac{\varepsilon}{2}C_6(4C_6)^{\frac{\alpha}{2}}\}\|\dot{V}(t)\|_q^{\alpha+2}\\&- \frac{\varepsilon}{4}\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2}.\end{aligned}$$

So, by choosing ε small enough in the previous inequality, we get for all $t \geq T$

$$K'(t) \leq -\frac{1}{2}\|\dot{U}(t)\|_p^{\alpha+2} - \frac{1}{2}\|\dot{V}(t)\|_q^{\alpha+2} - \frac{\varepsilon}{4}\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2}. \tag{3.6}$$

The inequality (3.6) shows that K is non increasing on $[T,+\infty[$. Moreover, from (1.11) and (1.6), we deduce

$$\lim_{t\to\infty} K(t) = 0.$$

If there exists $t_0 \geq T$ such that $K(t_0) = 0$, then $K(t) = 0$ for all $t \geq t_0$ and by using inequality (3.6) we obtain $\dot{U}(t) = \dot{V}(t) = 0$ for all $t \geq t_0$. This implies that (U, V) is stationary and then convergent solution. If it is not the case, we assume in the next that

$$\forall t \geq T, K(t) > 0, \tag{3.7}$$

and we will prove that $t \to \|\dot{U}(t)\|_p$ is integrable on $[T,+\infty[$. We have $\alpha < \dfrac{\theta}{1-\theta}$, then

$$\beta := \theta - (1-\theta)\alpha > 0.$$

From (3.7), we have for all $t \geq T$

$$-\frac{1}{\beta}(K(t)^{\beta})' = \frac{-K'(t)}{\{K(t)^{1-\theta}\}^{1+\alpha}}. \tag{3.8}$$

Since the following elementary inequality

$$\forall a, b \in \mathbb{R}_+, (a+b)^{\lambda} \leq 2(a^{\lambda} + b^{\lambda})$$

holds for any $\lambda \in [0, 2]$, then by using Cauchy-Schwarz inequality, we get for all $t \in \mathbb{R}_+$

$$\begin{aligned} K(t)^{1-\theta} \leq 4\{&\|\dot{U}(t)\|_p^{2(1-\theta)} + \|\dot{V}(t)\|_q^{2(1-\theta)} + |F(U(t))G(V(t))|^{1-\theta} \\ &+\|\dot{U}(t)\|_p^{1-\theta}\|G(V(t))\nabla F(U(t))\|_p^{(\alpha+1)(1-\theta)}\}. \end{aligned} \tag{3.9}$$

Once again by applying Young's inequality, we get

$$\|\dot{U}(t)\|_p^{1-\theta}\|G(V(t))\nabla F(U(t))\|_p^{(\alpha+1)(1-\theta)} \leq \|\dot{U}(t)\|_p^{\frac{1-\theta}{\theta}} + \|G(V(t))\nabla F(U(t))\|_p^{\alpha+1}.$$

Then, inequality (3.9) becomes

$$\begin{aligned} K(t)^{1-\theta} \leq 4\{&\|\dot{U}(t)\|_p^{2(1-\theta)} + \|\dot{V}(t)\|_q^{2(1-\theta)} + |F(U(t))G(V(t))|^{1-\theta} \\ &+\|\dot{U}(t)\|_p^{\frac{1-\theta}{\theta}} + \|G(V(t))\nabla F(U(t))\|_p^{\alpha+1}\}. \end{aligned} \tag{3.10}$$

Since $\theta \in]0, \frac{1}{2}]$, then $\dfrac{1-\theta}{\theta} \geq 1$ and $2(1-\theta) \geq 1$. Together with (3.5) implies that for all $t \geq T$ we have

$$\|\dot{U}(t)\|_p^{2(1-\theta)} \leq \|\dot{U}(t)\|_p, \|\dot{V}(t)\|_q^{2(1-\theta)} \leq \|\dot{V}(t)\|_q \text{ and } \|\dot{U}(t)\|_p^{\frac{1-\theta}{\theta}} \leq \|\dot{U}(t)\|_p.$$

So, from (3.10), we get

$$K(t)^{1-\theta} \leq 8\{\|\dot{U}(t)\|_p + \|\dot{V}(t)\|_q + |F(U(t))G(V(t))|^{1-\theta} + (C_1C_4)^{\alpha}\|G(V(t))\nabla F(U(t))\|_p\}. \tag{3.11}$$

Together (3.4) and the previous inequality implies that for all $t \geq T$ we have

(3.12)
$$K(t)^{1-\theta} \leq C_7\{\|\dot{U}(t)\|_p+\|\dot{V}(t)\|_q+|G(V(t))|^{1-\theta}\|\nabla F(U(t))\|_p+\|G(V(t))\nabla F(U(t))\|_p\}.$$

Where $C_7 = 8(1 + (C_1C_4)^\alpha + \frac{1}{C_\Gamma}) > 0$.
Now, Writing

$$\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2} = |G(V(t))|^{\theta(\alpha+2)}\{|G(V(t))|^{1-\theta}\|\nabla F(U(t))\|_p\}^{\alpha+2}.$$

From (3.3) we have

$$\forall t \geq T, |G(V(t))| \geq \delta.$$

Then, for all $t \geq T$

$$\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2} \geq \delta^{\theta(\alpha+2)}\{|G(V(t))|^{1-\theta}\|\nabla F(U(t))\|_p\}^{\alpha+2}.$$

Therefore, from (3.6) we have for all $t \geq T$

$$\begin{aligned}-K'(t) \geq & \frac{1}{2}\|\dot{U}(t)\|_p^{\alpha+2} + \frac{1}{2}\|\dot{V}(t)\|_q^{\alpha+2} + \frac{\varepsilon}{8}\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2} \\ & + \frac{\varepsilon}{8}\delta^{\theta(\alpha+2)}\{|G(V(t))|^{1-\theta}\|\nabla F(U(t))\|_p\}^{\alpha+2}.\end{aligned}$$

We assume, if it is necessary, that ε is small enough such that $C_8 := \dfrac{\varepsilon}{8}\delta^{\theta(\alpha+2)} \leq \dfrac{1}{2}$,
Thus, the previous inequality becomes

$$\begin{aligned}-K'(t) \geq C_8\{ & \|\dot{U}(t)\|_p^{\alpha+2} + \|\dot{V}(t)\|_q^{\alpha+2} + \{|G(V(t))|^{1-\theta}\|\nabla F(U(t))\|_p\}^{\alpha+2} \\ & + \|G(V(t))\nabla F(U(t))\|_p^{\alpha+2}\}.\end{aligned}$$

Since that $x \to x^{\alpha+2}$ is a convex function on $\mathbb{R}_+$, then

$$\begin{aligned}-K'(t) \geq & \frac{C_8}{4^{\alpha+1}}\{\|\dot{U}(t)\|_p + \|\dot{V}(t)\|_q + |G(V(t))|^{1-\theta}\|\nabla F(U(t))\|_p \\ & + \|G(V(t))\nabla F(U(t))\|_p\}^{\alpha+2}.\end{aligned}$$

Together with (3.8) and (3.12), implies that for every $t \geq T$ and $C_9 := \dfrac{\beta C_8}{(4C_7)^{\alpha+1}}$,
we have

$$-\frac{(K(t)^\beta)'}{C_9} \geq \|\dot{U}(t)\|_p + \|\dot{V}(t)\|_q + \{|G(V(t))| + |G(V(t))|^{1-\theta}\}\|\nabla F(U(t))\|_p. \tag{3.13}$$

Therefore, by integrating over any interval of time $[T, \tau]$ we get

$$\int_T^\tau \|\dot{U}(s)\|_p ds + \int_T^\tau \|\dot{V}(s)\|_q ds \leq \frac{1}{C_9}E(T)^\beta < \infty. \tag{3.14}$$

Consequently, the application $t \to \|\dot{U}(t)\|_p+\|\dot{V}(t)\|_q$ is integrable on $[T, +\infty[$. From this derives the convergence of (U, V) to an equilibrium point $(a, b) \in W(U, V) \subset S_F \times \mathbb{R}^q$.

4. Rate of convergence: Proof of Theorem 4

Based on what we have in the proof of Theorem 3, we shall establish some suitable differential inequalities. Next, by using the following Lemma (1), we obtain the desired estimate.

Lemma 1. *([15]) Let f be a positive solution of the following differential inequality*

$$f'(t) + Cf(t)^{\gamma} \leq 0, \forall t \geq 0.$$

If $C > 0$ and $\gamma \geq 1$, then we have

$$f(t) \leq \begin{cases} f(0)e^{-\beta t}, \forall t \geq 0, \\ (\frac{1}{C(\gamma-1)}t)^{-\frac{1}{\gamma-1}}, \forall t > 0. \end{cases} \tag{4.1}$$

Let us gather some facts from the Proof of Theorem 3. We showed in (3.13) that there exists T and a positive constant C_9 such that for all $t \geq T$, we have

$$-\frac{(K(t)^{\beta})'}{C_9} \geq \|\dot{U}(t)\|_p + \|\dot{V}(t)\|_q + \{|G(V(t))| + |G(V(t))|^{1-\theta}\}\|\nabla F(U(t))\|_p. \tag{4.2}$$

From (3.12), we have

$K(t)^{1-\theta} \leq C_7\{\|\dot{U}(t)\|_p+\|\dot{V}(t)\|_q+|G(V(t))|^{1-\theta}\|\nabla F(U(t))\|_p+\|G(V(t))\nabla F(U(t))\|_p\}.$

Together with (4.2), gives

$$\forall t \geq T, (K(t)^{\beta})' \leq -\frac{C_9}{C_7}K(t)^{1-\theta} = -\frac{C_9}{C_7}(K(t)^{\beta})^{\frac{1-\theta}{\beta}}.$$

Let

$$f(t) = K(t)^{\beta}, C = \frac{C_9}{C_7}, \gamma = \frac{1-\theta}{\beta} > 1 \text{ and } C_{10} = (\frac{1}{C(\gamma-1)})^{-\frac{1}{\gamma-1}}.$$

Then, by applying the Lemma 1 we obtain

$$\forall t \geq T, K(t)^{\beta} \leq C_{10}t^{-\dfrac{\beta}{1-\theta-\beta}}. \tag{4.3}$$

Now, by writing

$$\|U(t)-a\|_p + \|V(t)-b\|_q \leq \int_t^{\infty}(\|\dot{U}(s)\|_p + \|\dot{V}(s)\|_q)ds,$$

and using (4.2), we get

$$\forall t \geq T, \|U(t)-a\|_p + \|V(t)-b\|_q \leq \frac{1}{C_9}K(t)^{\beta}.$$

Together with (4.3), implies

$$\forall t \geq T, \|U(t)-a\|_p + \|V(t)-b\|_q \leq \frac{C_{10}}{C_9}t^{-\dfrac{\beta}{1-\theta-\beta}}. \tag{4.4}$$

Similarly, we will estimate the speed of decay for the damping term. For that, by using of the first equation of (1.1) we get

$$\|\ddot{U}(t)\|_p \leq \|\dot{U}(t)\|_p^{\alpha+1} + |G(V(t))| \|\nabla F(U(t))\|_p.$$

From (3.5) we have

$$\forall t \geq T, \|\dot{U}(t)\|_p^{\alpha+1} \leq \|\dot{U}(t)\|_p.$$

Then

$$\|\ddot{U}(t)\|_p \leq \|\dot{U}(t)\|_p + |G(V(t))| \|\nabla F(U(t))\|_p.$$

Thanks to (4.2), we have

$$\forall t \geq T, \|\ddot{U}(t)\|_p \leq -\frac{1}{C_9}(K(t)^\beta)'. \tag{4.5}$$

We have $\lim_{t\to+\infty} \|\dot{U}(t)\|_p = 0$, see (1.11), then

$$\forall t \in \mathbb{R}_+, \dot{U}(t) = -\int_t^{+\infty} \ddot{U}(s) ds.$$

So, by using (4.5) and (4.3), we obtain for all $t \geq T$

$$\|\dot{U}(t)\|_p \leq \int_t^{+\infty} \|\ddot{U}(s)\|_p ds \leq \frac{1}{C_9} K(t)^\beta \leq \frac{C_{10}}{C_9} t^{-\dfrac{\beta}{1-\theta-\beta}}. \tag{4.6}$$

It remains now to estimate the speed of decay for $\|\dot{V}(t)\|_q$. Together the second equation of (1.1) and (3.5) implies

$$\forall t \geq T, \|\ddot{V}(t)\|_q \leq \|\dot{V}(t)\|_q + \|F(U(t))\nabla G(V(t))\|_q.$$

In order to develop calculation in the same way as previous, we shall establish some new estimates for the term $\|F(U(t))\nabla G(V(t))\|_q$ who appeared in the previous inequality. For that, we introduce a new Lyapunov function

$$L(t) = K(t) - \varepsilon \|F(U(t))\nabla G(V(t))\|_q^\alpha \langle F(U(t))\nabla G(V(t)), \dot{V}(t)\rangle_q,$$

which is obtained by perturbation of K with the following mixed term

$$N(t) := \|F(U(t))\nabla G(V(t))\|_q^\alpha \langle F(U(t))\nabla G(V(t)), \dot{V}(t)\rangle_q.$$

By differentiating $N(t)$ on all points such that $F(U)\nabla G(V)$ is not zero, we get

$$\begin{aligned}
N'(t) &= \alpha\|F(U(t))\nabla G(V(t))\|_q^{\alpha-2}\langle \nabla F(U(t)), \dot U(t)\rangle_p \langle F(U(t))\nabla G(V(t)), \dot V(t)\rangle_q \\
&\quad \langle F(U(t))\nabla G(V(t)), \nabla G(V(t))\rangle_q + \alpha\|F(U(t))\nabla G(V(t))\|_q^{\alpha-2} \\
&\quad \langle F(U(t))\nabla G(V(t)), F(U(t))\nabla^2 G(V(t)).\dot V(t)\rangle_q \langle F(U(t))\nabla G(V(t)), \dot V(t)\rangle_q \\
&\quad + \|F(U(t))\nabla G(V(t))\|_q^{\alpha}\langle \nabla F(U(t)), \dot U(t)\rangle_p \langle \nabla G(V(t)), \dot V(t)\rangle_q \\
&\quad + F(U(t))\|F(U(t))\nabla G(V(t))\|_q^{\alpha}\langle \nabla^2 G(V(t)).\dot V(t), \dot V(t)\rangle_q \\
&\quad + \|F(U(t))\nabla G(V(t))\|_q^{\alpha+2} - F(U(t))\|F(U(t))\nabla G(V(t))\|_q^{\alpha}\|\dot V(t)\|_q^{\alpha} \\
&\quad \langle \nabla G(V(t)), \dot V(t)\rangle_q.
\end{aligned}$$

Let

$$C_{11} = \sup_{t\in R_+} |F(U(t))| \text{ and } C_{12} = \sup_{t\in R_+} \|\nabla^2 G(V(t))\|_{q,q}.$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
-\varepsilon N'(t) &\le -\varepsilon\|F(U(t))\nabla G(V(t))\|_q^{\alpha+2} + \frac{\alpha\varepsilon}{2}C_1C_2\{\|\dot U(t)\|_p^2 + \|\dot V(t)\|_q^2\} \\
&\quad + (\alpha+1)\varepsilon C_{10}C_{11}\|F(U(t))\nabla G(V(t))\|_q^{\alpha}\|\dot V(t)\|_q^2 \\
&\quad + \frac{\varepsilon}{2}C_1C_2\|F(U(t))\nabla G(V(t))\|_q^{\alpha}\{\|\dot U(t)\|_p^2 + \|\dot V(t)\|_q^2\} \\
&\quad + \frac{\varepsilon}{2}C_{10}\|F(U(t))\nabla G(V(t))\|_q^{\alpha}\|\dot V(t)\|_q^{\alpha}\{\|\nabla G(V(t))\|_q^2 + \|\dot V(t)\|_q^2\}.
\end{aligned}$$

By similar computations to those that provided the inequality (3.6) and by choosing ε small enough as for the inequality, we have

$$\begin{aligned}
L'(t) &\le -\frac{1}{2}\|\dot U(t)\|_p^{\alpha+2} - \frac{1}{2}\|\dot V(t)\|_q^{\alpha+2} - \frac{\varepsilon}{4}\|G(V(t))\nabla F(U(t))\|_p^{\alpha+2} \\
&\quad -\frac{\varepsilon}{4}\|F(U(t))\nabla G(V(t))\|_q^{\alpha+2}.
\end{aligned} \tag{4.7}$$

Same arguments as in the proof of Theorem 3, the previous estimate holds true for those points which satisfies $F(U)\nabla G(V) = 0$. In the same way we establish a similar inequality to (3.12). So, there exists an instant which is denoted also by T such that for all $t \ge T$

$$\begin{aligned}
L(t)^{1-\theta} &\lesssim \|\dot U(t)\|_p + \|\dot V(t)\|_q + \{|G(V(t))| + |G(V(t))|^{1-\theta}\}\|\nabla F(U(t))\|_p \\
&\quad +\|F(U(t))\nabla G(V(t))\|_q.
\end{aligned} \tag{4.8}$$

By similar computations as for the inequality (3.13), we obtain for all $t \ge T$

$$\begin{aligned}
-(L(t)^{\beta})' &\gtrsim \|\dot U(t)\|_p + \|\dot V(t)\|_q + \{|G(V(t))| + |G(V(t))|^{1-\theta}\}\|\nabla F(U(t))\|_p \\
&\quad +\|F(U(t))\nabla G(V(t))\|_q.
\end{aligned} \tag{4.9}$$

Together (4.8) and (4.9), gives us

$$-(L(t)^{\beta})' \gtrsim L(t)^{1-\theta} = (L(t)^{\beta})^{\frac{1-\theta}{\beta}}.$$

Thanks to the Lemma 1, we have

$$\forall t \geq T, L(t)^{\beta} \lesssim t^{-\dfrac{\beta}{1-\theta-\beta}}. \tag{4.10}$$

From previous, we have for all $t \geq T$

$$\begin{aligned} \|\dot{V}(t)\|_q &\leq \int_t^{+\infty} \|\ddot{V}(s)\|_q ds \\ &\leq \int_t^{+\infty} \{\|\dot{V}(s)\|_q + \|F(U(t))\nabla G(V(t))\|_q\} ds \\ &\lesssim -\int_t^{+\infty} (L(s)^{\beta})' ds \lesssim L(t)^{\beta}. \end{aligned}$$

Since we have (4.10), then

$$\forall t \geq T, \|\dot{V}(t)\|_q \lesssim t^{-\dfrac{\beta}{1-\theta-\beta}}. \tag{4.11}$$

To finish now, let us gather (4.4), (4.6) and (4.11), then by changing constants if necessary, results stated in Theorem 3 are completely proved for every $t \in \mathbb{R}_+$.

5. Non-convergence result

From Theorem 3 we derive the following result

Corollary 2. *Suppose that the assumptions of the Corollary 1 holds. In addition, we suppose that F satisfies* (1.4) *with a uniform Lojasiewicz exponent $\theta = \frac{1}{2}$. Then, for all $\alpha \in [0, 1[$, every global and bounded solution (U, V) of problem* (1.1) *converges to an equilibrium point $(a, b) \in S_F \times \mathbb{R}^q$.*

In this section, we are interested in problem (1.1) when $\alpha = 1$. Precisely, we show the existence of non convergent solution even if the assumptions stated in Theorem 3 hold. For that, we make use of the following non convergence result:

Theorem 5. *(see* [13]*)*
Let f be a locally Lipshitz function defined on $\mathbb{R}$ such that

$$\begin{cases} f(x) < 0 \text{ for } x < a, \\ f(x) = 0 \text{ for } a \leq x \leq b, \\ f(x) > 0 \text{ for } b < x. \end{cases} \tag{5.1}$$

Then, for every non constant and bounded solution of the following ordinary differential equation

$$\ddot{y}(t) + |\dot{y}(t)|\dot{y}(t) + f(y(t)) = 0, t \in \mathbb{R}_+.$$

There exist sequences $t_n \to +\infty$ and $s_n \to +\infty$ such that $y(t_n) < a$ and $y(s_n) > b$ for all $n \in \mathbb{N}$.

Let us define the following functions

$$F(x) = \begin{cases} -(x-a)^2, \text{if } x \leq a, \\ (x-b)^2, \text{if } b \leq x, \\ 0, \text{if } x \in [a,b]. \end{cases}$$

and

$$G(x) = \begin{cases} e^{(x-a)^4}, \text{if } x \leq a, \\ e^{(x-b)^4}, \text{if } b \leq x, \\ 1, \text{if } x \in [a,b]. \end{cases}$$

The function F satisfies assumptions (1.6) and (1.4) with $\theta = \dfrac{1}{2}$. Also, the function G satisfies (1.16) with $\delta = 1$. We consider the following gradient like system

$$\begin{cases} \ddot{u}(t) + |\dot{u}(t)|\dot{u}(t) + G(v(t))(F')(u(t)) = 0, \\ \ddot{v}(t) + |\dot{v}(t)|\dot{v}(t) + F(u(t))G'(v(t)) = 0, \\ t \in \mathbb{R}_+. \end{cases} \tag{5.2}$$

The Theorem 5 shows that every component of a non constant and global solution of (5.2) does not converge.

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